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Numerical Solution of the Constrained Re-entry Vehicle Trajectory Problem via Quasilinearization

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The effectiveness of three quasilinearization algorithms in solving variations of a complex re-entry vehicle problem was investigated. These algorithms were developed to solve classes of constrained optimal control problems without using penalty functions. The constraints are control bound constraints and/or inequality constraints on functions of the state and control variables. In applying these methods, there is no need to guess the sequence of constrained and unconstrained arcs. All this information will be determined within the iterative procedure using necessary conditions for the optimal control. It is shown that convergence to a desired accuracy was achieved within a single-digit number of iterations. As the result of the small convergence region, it was necessary to use an unconstrained optimal re-entry vehicle trajectory as the initial nominal function for the constrained problem. It has not been possible to compare the convergence sensitivities of the two methods to optimal control problems with control bounds. Other computational characteristics of these two methods are basically similar.

Nomenclature

A = time-varying 1×7 matrix
B = time-varying function
C_D = drag coefficient
C_L = lift coefficient
D = drag force
E = time-varying 7×7 matrix
F = time-varying 7×1 matrix
g = acceleration due to gravity
g_0 = acceleration due to gravity at sea level (32.172 ft/s)
H = Hamiltonian
H' = auxiliary Hamiltonian
h = altitude
J = performance index
L = lift force
m = vehicle mass
N = quasilinearization iteration number
R = radius of the Earth (20,904,000 ft)
r = flight-path angle relative to local horizontal
S = reference area
t_f = final time
V = velocity
X = constraint function of state and control variables
X' = auxiliary constraint function associated with X
x = surface range
Y = column vector
y = convergent vector
α = vehicle angle of attack
α' = 'relaxed' control associated with control variable
λ_1 = adjoint variable associated with state variable V
λ_2 = adjoint variable associated with state variable r
λ_3 = adjoint variable associated with state variable h
λ_4 = adjoint variable associated with state variable x
δ = small preselected positive quantity
ρ = error term
σ = Lagrange multiplier associated with upper bound of control variable
τ = Lagrange multiplier associated with lower bound of control variable
μ = Lagrange multiplier associated with constraint function X

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I. Introduction

PREVIOUS computational experience indicates that quasilinearization is one of the more feasible numerical methods for solving complicated dynamic optimization problems in aerospace systems.[1-6] However, it appears that aerospace trajectory optimization problems with inequality constraints on functions of the state and control variables have not been solved via quasilinearization without using penalty functions. Many practical problems have control bounds and inequality constraints on functions of the state and control variables. The re-entry vehicle trajectory problem of Sec. II is an example.

To the best knowledge of the authors, there are three effective quasilinearization algorithms which could be used to solve optimal control problems with control bounds.[4-9] In Ref. 3, a quasilinearization method is used to solve problems whose optimal solutions are of "bang-bang" control. A variation of the re-entry vehicle trajectory problem has been tackled by Leondes and Paine.[4-6] It was found that the region of convergence is quite small. The method of Leondes and Paine[4-6] and the method of Yeo et al.[7] use the same criterion to determine whether the control is on the boundary at any point of the nominal trajectory. In addition to its ability to solve a wider class of optimal control problems with control bounds, it has been shown that the method proposed by Yeo[8,9] has a larger region of convergence than the method of Yeo et al.[7] in solving two simple problems. This is the distinct advantage of the method proposed by Yeo[8,9] over the other two methods.[4-7]

The method described in Ref. 7 has been extended to handle a class of optimal control problems with control bounds and inequality constraints on functions of the state and control variables. It is perhaps the only published quasilinearization method which could be used to solve this class of problems without using penalty function. The proposed quasilinearization algorithm[10] displays the following advantages. First, convergence is rapid. Secondly, no prior information is required regarding the sequence of constrained and unconstrained arcs and the constraints which are on their boundaries along a specified constrained arc of the optimal trajectory. All this information will be determined within the iterative procedure using the necessary conditions for optimal control. Thirdly, at any point along the nominal trajectory, only those inequality constraints which are on their boundaries play a part in the solution. Finally, the optimal solution of the problem is obtained as no penalty function is used. However, only relatively simple problems have been solved.[10,11]

The solution of the re-entry vehicle trajectory problem[4-6,12,13] is an appropriate choice for a more complex problem. It is well-known that this problem is computationally difficult because of intergration instability and the sensitivity of the adjoint variables.[4-6] The major objectives of this investigation are to answer the following questions.

1) Are the methods proposed in Refs. 7-9 effective in solving the re-entry vehicle trajectory problem with control bounds? Furthermore, which is the more effective method with regard to the following characteristics: simplicity of formulation and implementation, computer storage requirements, convergence sensitivity, and convergence time.

2) Is the quasilinearization algorithm of Yeo[10] an effective method for solving the re-entry vehicle trajectory problem with deceleration limitation or both deceleration limitation and control bounds?

II. Re-entry Vehicle Trajectory Problem [4-6,12,13]

As a spacecraft approaches the Earth during the return flight, the Earth's attraction increases as the craft moves nearer. The trajectory of the craft between the time it enters the atmosphere and the time a horizontal flight path is attained is the trajectory of main concern here. Owing to the physiological effect of high deceleration on the astronaut, limitation is imposed on the spacecraft deceleration. Such a constraint can be represented by the following inequality:

$$(L^2+D^2)^{1/2}/m \leq 5g_0 \tag{1}$$

The problem considered is to minimize

$$J = -V(t_f) \tag{2}$$

subject to

1) The state equations:

$$\dot{V} = -D/m - g\sin r \qquad V(0) = 36{,}000 \text{ ft/s} \tag{3}$$

$$\dot{r} = \frac{L}{mV} + \left(\frac{V}{R+h} - \frac{g}{v}\right)\cos r \qquad r(0) = -7.5 \text{ deg}, \quad r(t_f) = 0 \text{ deg} \tag{4}$$

$$\dot{h} = V\sin r \qquad h(0) = 400{,}000 \text{ ft} \tag{5}$$

$$\dot{x} = V\left(\frac{\cos r}{1+h/R}\right) \qquad x(0) = 0 \tag{6}$$

2) An inequality constraint:

Equation (1) can be rearranged to obtain the following expression:

$$X(h,V,\alpha) = (L^2+D^2)^{1/2}/mg_0 - 5 \leq 0 \tag{7}$$

3) The control bound constraints:

$$-22.5 \text{ deg} \leq \alpha \leq 22.5 \text{ deg} \tag{8}$$

where

$$g = g_0 R^2/(R+h)^2$$

$$D = C_D(\alpha)\rho(h)V^2(S/2)$$

$$L = C_L(\alpha)\rho(h)V^2(S/2)$$

$$\rho(h) = 0.002378\, e^{-\beta h} \text{ slug/ft}^3$$

$$\beta = 1/22{,}000 \text{ ft}^{-1}$$

$$C_D(\alpha) = 0.27 + 1.82\sin^2\alpha$$

$$C_L(\alpha) = 0.6\sin 2\alpha$$

$$S/2m = 0.26245 \text{ ft}^2/\text{slug}$$

$$t_f = 300 \text{ s}$$

III. Optimal Control Problem with Control Bounds

The optimal control problem under consideration is to minimize Eq. (2) subject to the conditions in Eqs. (3-6) and (8). This problem could be solved by either the method of Yeo et al.[7] or the method of Yeo.[8,9] For both methods, conditions used to determine whether the control variable α is on its bound at any point of the nominal trajectory are derived later. At the $(N+1)$st stage of the iteration, the computed value of α_{N+1} was set equal to 22.5 deg if it is greater than 22.5 deg, and equal to -22.5 deg if it is smaller than -22.5 deg.

Method I[7]

Define

$$H = \lambda_1\left(-\frac{D}{m} - g\sin r\right) + \lambda_2\left[\frac{L}{mV} + \left(\frac{V}{R+h} - \frac{g}{V}\right)\cos r\right] + \lambda_3 V\sin r + \lambda_4 V\left(\frac{\cos r}{1+h/R}\right) \tag{9}$$

Along the optimal trajectory, the following conditions must be satisfied:

1) When $-22.5 \text{ deg} < \alpha < 22.5 \text{ deg}$, $\partial H/\partial\alpha = 0$ gives

$$\alpha' = \alpha = \tfrac{1}{2}\tan^{-1}\left(\frac{0.6}{0.91}\cdot\frac{\lambda_2}{\lambda_1 V}\right) \qquad (-45 \text{ deg} \leq \alpha \leq 45 \text{ deg}) \tag{10}$$

2) When $\alpha = -22.5$ deg,

$$\alpha' = \tfrac{1}{2}\tan^{-1}\left(\frac{0.6}{0.91}\cdot\frac{\lambda_2}{\lambda_1 V}\right) \leq -22.5 \text{ deg} \qquad (-45 \text{ deg} \leq \alpha' \leq 45 \text{ deg}) \tag{11}$$

3) When $\alpha = 22.5$ deg,

$$\alpha' = \tfrac{1}{2}\tan^{-1}\left(\frac{0.6}{0.91}\cdot\frac{\lambda_2}{\lambda_1 V}\right) \geq 22.5 \text{ deg} \qquad (-45 \text{ deg} \leq \alpha' \leq 45 \text{ deg}) \tag{12}$$

At the $(N+1)$st stage of the iteration, conditions (10-12) were used to determine whether a point along the nominal trajectory is on a control bound or within the control bounds using values of λ_1, λ_2, and V at the Nth stage of the iteration.

Method II[8,9]

Define

$$H = \lambda_1\left(-\frac{D}{m} - g\sin r\right) + \lambda_2\left[\frac{L}{mV} + \left(\frac{V}{R+h} - \frac{g}{V}\right)\cos r\right] + \lambda_3 V\sin r + \lambda_4 V\left(\frac{\cos r}{1+h/R}\right) + \sigma(\alpha - 22.5 \text{ deg}) + \tau(-22.5 \text{ deg} - \alpha) \tag{13}$$

Along the optimal trajectory, the following conditions must be satisfied:

1) When $-22.5 \text{ deg} < \alpha < 22.5 \text{ deg}$, $\sigma = 0$, $\tau = 0$, and $\partial H / \partial \alpha = 0$ gives

$$\alpha = \tfrac{1}{2} \tan^{-1}\left(\frac{0.6}{0.91} \cdot \frac{\lambda_2}{\lambda_1 V}\right) \qquad (-45 \text{ deg} \leq \alpha \leq 45 \text{ deg}) \tag{14}$$

2) When $\alpha = 22.5$ deg, $\partial H / \partial \alpha = 0$ gives

$$\sigma = \lambda_1 \frac{S\rho V^2}{m} 0.9 \sin 2\alpha - \lambda_2 \frac{S\rho V}{m} 0.6 \cos 2\alpha > 0 \tag{15}$$

3) When $\alpha = -22.5$ deg, $\partial H / \partial \alpha = 0$ gives

$$\tau = -\lambda_1 \frac{S\rho V^2}{m} 0.9 \sin 2\alpha + \lambda_2 \frac{S\rho V}{m} 0.6 \cos 2\alpha > 0 \tag{16}$$

At the $(N+1)$st stage of the iteration, a point along the nominal trajectory was taken:
1) On the control bound $\alpha = 22.5$ deg, if

$$\alpha_N \geq 22.5 \text{ deg} \tag{17}$$

and

$$\sigma = \lambda_{1(N)} \frac{S\rho_N V_N^2}{m} 0.9 \sin 45 \text{ deg} - \lambda_{2(N)} \frac{S\rho_N V_N}{m} 0.6 \cos 45 \text{ deg} > 0 \tag{18}$$

2) On the control bound $\alpha = -22.5$ deg, if

$$\alpha_N \leq -22.5 \text{ deg} \tag{19}$$

and

$$\tau = -\lambda_{1(N)} \frac{S\rho_N V_N^2}{m} 0.9 \sin 45 \text{ deg} + \lambda_{2(N)} \frac{S\rho_N V_N}{m} 0.6 \cos 45 \text{ deg} > 0 \tag{20}$$

3) Within the control bounds if neither conditions (17) and (18) nor conditions (19) and (20) are satisfied.

IV. Optimal Control Problem with an Inequality Constraint and Control Bounds

The optimal control problem under consideration is to minimize Eq. (2) subject to the conditions in Eqs. (3-8). In the quasilinearization algorithm of Yeo,[10] an auxiliary minimization problem which minimizes Eq. (2) subject to the conditions in Eqs. (3-6) is considered. This auxiliary minimization problem differs from the original problem in that the inequality constraints (7) and (8) are assumed absent. We define

$$H' = \lambda_1\left(-\frac{D}{m} - g \sin r\right) + \lambda_2\left[\frac{L}{mV} + \left(\frac{V}{R+h} - \frac{g}{V}\right)\cos r\right] + \lambda_3 V \sin r + \lambda_4 V\left(\frac{\cos r}{1+h/R}\right) \tag{21}$$

The 'relaxed' control α' of the original control problem is

$$\alpha' = \tfrac{1}{2} \tan^{-1}\left(\frac{0.6}{0.91} \cdot \frac{\lambda_2}{\lambda_1 V}\right) \qquad (-45 \text{ deg} \leq \alpha' \leq 45 \text{ deg}) \tag{22}$$

which is obtained from $H'_\alpha = 0$ after substituting $\alpha = \alpha'$. Let

$$U = \begin{cases} 22.5 \text{ deg} & \text{if } \alpha' \geq 22.5 \text{ deg} \\ -22.5 \text{ deg} & \text{if } \alpha' \leq -22.5 \text{ deg} \\ \alpha' & \text{if } -22.5 \text{ deg} < \alpha' < 22.5 \text{ deg} \end{cases} \tag{23}$$

and

$$X' = X(h, V, U) \tag{24}$$

Along the optimal trajectory of the original re-entry vehicle trajectory problem, a point is
1) Within any constrained boundary if

$$22.5 \text{ deg} > \alpha' > -22.5 \text{ deg} \quad \text{and} \quad X' < 0 \tag{25}$$

2) On the boundary $X = 0$ if

$$X' \geq 0 \tag{26}$$

3) On the control bound $\alpha = 22.5$ deg if

$$\alpha' \geq 22.5 \text{ deg} \quad \text{and} \quad X' < 0 \tag{27}$$

4) On the control bound $\alpha = -22.5$ deg if

$$\alpha' \leq -22.5 \text{ deg} \quad \text{and} \quad X' < 0 \tag{28}$$

At the $(N+1)$st stage of the iteration, conditions in Eqs. (22-28) using values of λ_1, λ_2, V, and h at the Nth stage of the iteration were used to determine whether a point along the nominal trajectory was on any constraint boundary of Eqs. (7) and (8). The computed value of α_{N+1} was set equal to 22.5 deg if it is greater than 22.5 deg, and equal to -22.5 deg if it is less than -22.5 deg.

V. Optimal Control Problem with an Inequality Constraint

The optimal control problem under consideration is to minimize Eq. (2) subject to the conditions in Eqs. (3-7). As it is a simplified version of the optimal control problem of Sec. IV, the quasilinearization algorithm of Yeo[10] was employed to handle this problem after making some straightforward simplifications.

Along the optimal re-entry vehicle trajectory of the original problem, a point is
1) On the constraint boundary $X = 0$ if

$$X' = X(h, V, \alpha') \geq 0 \tag{29}$$

where α' is defined in Eq. (22), and
2) Within the constrained boundary if

$$X' < 0 \tag{30}$$

At the $(N+1)$st stage of the iteration, conditions in Eqs. (22), (29), and (30) using values of λ_1, λ_2, V, and h at the Nth stage of the iteration were used to determine whether a point along the nominal trajectory was on the constraint boundary $X = 0$.

VI. Numerical Solution

Three variations of the re-entry vehicle trajectory problem, namely problems with 1) control bound constraint, 2) inequality constraint, and 3) both control bound and inequality constraints, were solved numerically with appropriate quasilinearization algorithms.[7-10] A HP3000 digital computer and double precision arithmetic were employed to solve these three variations and the unconstrained version described in the Appendix. All the algorithms were programmed in Fortran IV. The interval of integration was divided into 100 steps. The first-order differential equation systems were integrated using a fourth-order Runge Kutta integration procedure. Systems of linear algebraic equations were solved using the Gauss Elimination Method. For the purpose of numerical solution, the values of the variables were normalized using 10^6 ft = 1 ft′, 10^2 s = 1 s′; and 10^{12} slug = 1 slug′, where the prime denotes a normalized unit. In this study, linearized differential equations of all algorithms were solved using the proposed method of Ref. 1.

Choice of Initial Nominal Functions

As the result of a small region of convergence for all three variations of the re-entry vehicle trajectory problem, a converged result could not be obtained using any of the proposed quasilinearization algorithms[7-10] with many sets of initial nominal functions. Converged results of the three variations of the re-entry vehicle trajectory problem were eventually generated using optimal solution of the unconstrained problem as the initial nominal function. Details of the numerical solution of the unconstrained re-entry vehicle trajectory problem are given in the Appendix. For solution of problems involving an inequality constraint on a function of the state and control variables, $\mu(t)=0$ was used as the initial nominal function. The initial nominal functions $\alpha(t)$, the sensed deceleration, and $h(t)$ were displayed as dotted lines in Figs. 1-3.

Convergence Criterion

Convergence was defined as having been reached when

$$\rho < 5 \times 10^{-3} \tag{31}$$

where

$$\begin{aligned}\rho = &\max_{t\in[0,t_f]} |V_{N+1}(t) - V_N(t)| \\ &+ \max_{t\in[0,t_f]} |r_{N+1}(t) - r_N(t)| + \max_{t\in[0,t_f]} |h_{N+1}(t) - h_N(t)| \\ &+ \max_{t\in[0,t_f]} |x_{N+1}(t) - x_N(t)| \\ &+ \sum_{i=1}^{3} \max_{t\in[0,t_f]} |\lambda_{i(N+1)} - \lambda_{i(N)}| + \max_{t\in[0,t_f]} |\alpha_{N+1}(t) - \alpha_N(t)|\end{aligned} \tag{32}$$

for the problem without the inequality constraint. For problems with the inequality constraint, an additional term

$$\max_{t\in[0,t_f]} |\mu_{N+1}(t) - \mu_N(t)|$$

was added to the right-hand side of Eq. (32)

Computational Results

Converged trajectories of the three variations of the re-entry vehicle trajectory problem are displayed on Figs. 1-4.

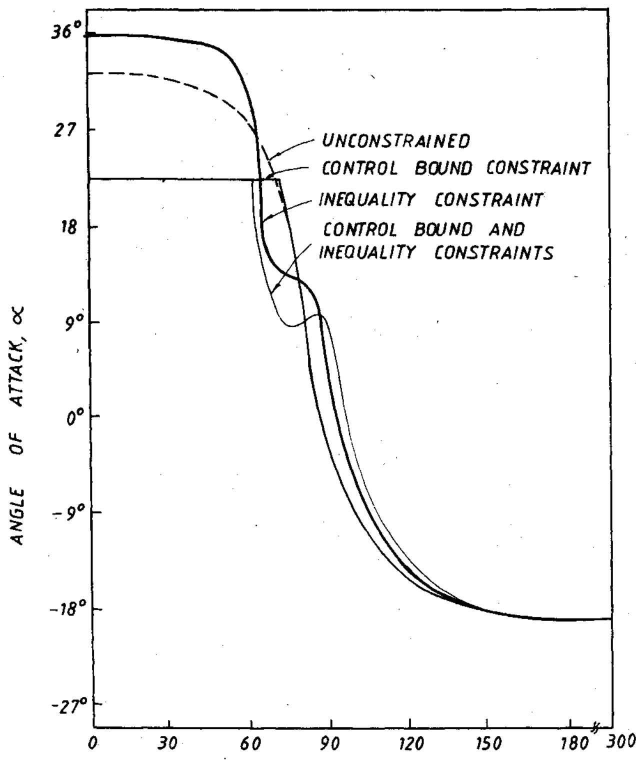

Fig. 1 Time histories of vehicle angle of attack α.

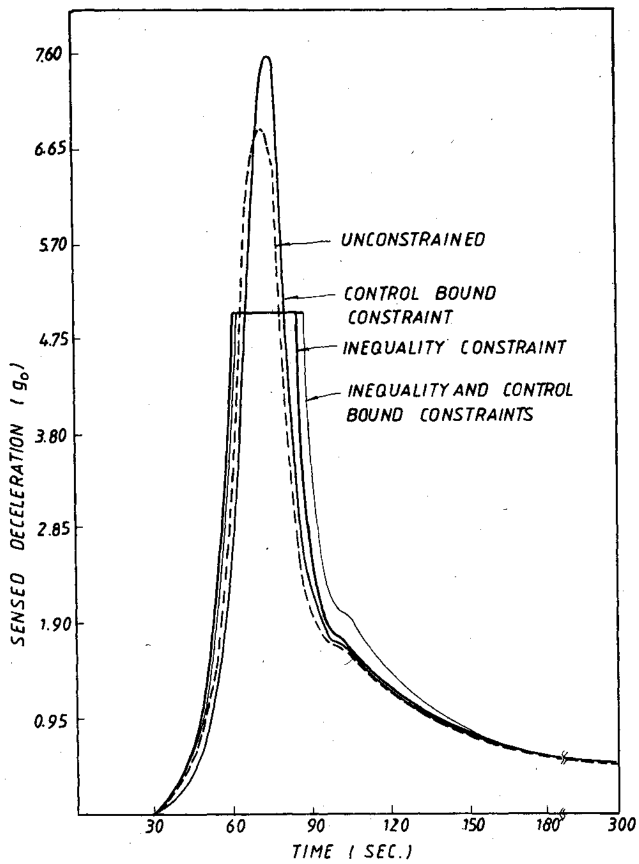

Fig. 2 Time histories of sensed deceleration.

Table 1 shows the convergence characteristics for the three variations of the re-entry vehicle trajectory problem. The convergence rate was obviously slowed down by the presence of the inequality constraint. It is interesting to note that CPU time per iteration is around 36 s and is independent of the

Table 1 Convergence characteristics

Convergence characteristics	Control bounds Method I	Control bounds Method II	Inequality constraint	Both
CPU time, s	107	108	217	289
Number of iterations at convergence	3	3	6	8
CPU time per iteration, s	36.0	35.6	36.1	36.1

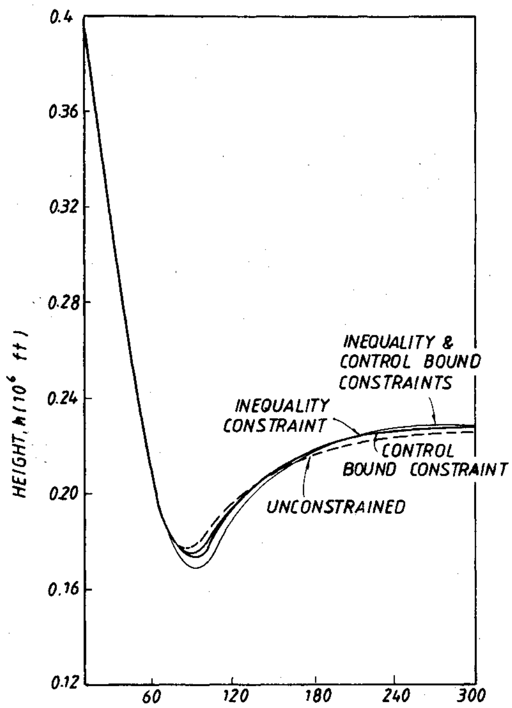

Fig. 3 Time histories of vehicle altitude h.

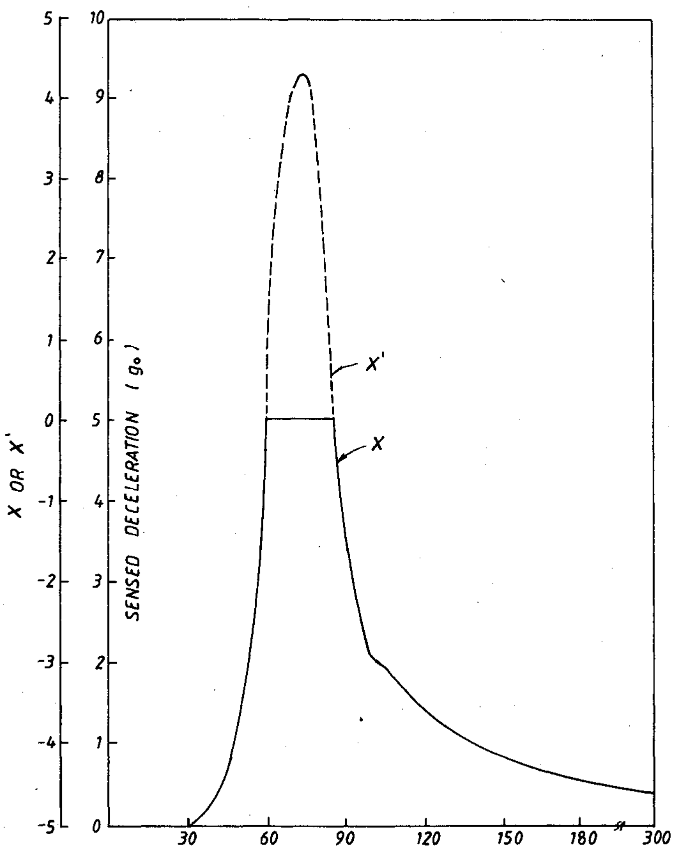

Fig. 4 Time histories of X and X' for a re-entry vehicle trajectory problem with both control bounds and deceleration limitation.

quasilinearization algorithm or type of constraint. For the re-entry vehicle trajectory problem with control bounds, it is obvious from Table 1 that there is little difference in the rate of convergence of the two methods.

VII. Conclusion

The investigation detailed in this paper produces the following conclusions:

1) Both the method proposed by Yeo[8,9] and the method of Yeo et al.[7] are applicable to solve the re-entry vehicle trajectory problem with control bounds. As the result of the small convergence region, it has not been possible to compare the convergence sensitivities of these two methods. Other computational characteristics of these two methods are basically similar.

2) The quasilinearization algorithm proposed by Yeo,[10] or its simplified version, could be used for numerical solution of the complex re-entry vehicle trajectory problem with both deceleration limitation and control bounds, or deceleration limitation alone, respectively. However, the region of convergence is small. In applying these algorithms, there is no need to guess the sequence of constrained or unconstrained arcs. All this information will be determined within the iterative procedure using the necessary conditions for optimal control. The optimal solution is obtained since no penalty function is used.

3) The convergence is rapid for the three variations of the problem under study. Rate of convergence was found to slow down considerably when an inequality constraint was imposed to limit the deceleration. CPU time per iteration is about the same regardless of the presence of an inequality constraint and/or control bounds.

4) The optimal trajectory of a complex unconstrained optimal control problem could be feasible initial nominal functions for numerical solution of the optimal control problem with control bounds and/or inequality constraint via quasilinearization.

Appendix: Optimal Solution for the Unconstrained Re-entry Vehicle Trajectory Problem

For optimal solution of the unconstrained re-entry vehicle trajectory problem, we define

$$H=\lambda_1\left(-\frac{D}{m}-g\,\sin r\right)+\lambda_2\left[\frac{L}{mV}+\left(\frac{V}{R+h}-\frac{g}{V}\right)\cos r\right]$$

$$+\lambda_3 V\sin r+\lambda_4 V\left(\frac{\cos r}{1+h/R}\right) \tag{A1}$$

Adjoint equations:

$$\dot\lambda_1=-\partial H/\partial V \qquad \lambda_1(t_f)=-1 \tag{A2}$$

$$\dot\lambda_2=-\partial H/\partial r \tag{A3}$$

$$\dot\lambda_3=-\partial H/\partial h \qquad \lambda_3(t_f)=0 \tag{A4}$$

$$\dot\lambda_4=-\partial H/\partial x=0 \qquad \lambda_4(t_f)=0 \tag{A5}$$

Conditions in Eq. (A5) imply

$$\lambda_4=0 \tag{A6}$$

Optimally condition:

$$\frac{\partial H}{\partial\alpha}=-\lambda_1\frac{S\rho V^2}{m}0.91\,\sin 2\alpha+\lambda_2\frac{S\rho V}{m}0.6\,\cos 2\alpha=0 \tag{A7}$$

The following relation can be derived from Eq. (A7):

$$\alpha=\tfrac{1}{2}\tan^{-1}\left(\frac{0.6}{0.91}\cdot\frac{\lambda_2}{\lambda_1 V}\right) \tag{A8}$$

Suppose that at the Nth stage of the iteration, an approximate solution reasonably close to the exact solution has been obtained. The linearized form of Eq. (A7) at the

$(N+1)$st stage of the iteration about the nominal functions, which are the solutions obtained at the Nth stage of the iteration, can be rearranged to give

$$\alpha_{N+1} = A(Y_N, \alpha_N) Y_{N+1} + B(Y_N, \alpha_N) \tag{A9}$$

where

$$Y = \begin{bmatrix} V \\ r \\ h \\ x \\ \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix}$$

After substituting Eq. (A9) into the linearized forms of Eqs. (3-6) and Eqs. (A2-A4) a linear differential system

$$\dot{Y}_{N+1} = E(Y_N, \alpha_N) Y_{N+1} + F(Y_N, \alpha_N) \tag{A10}$$

is obtained.

A step-by-step description of the quasilinearization algorithm for numerical solution of the unconstrained re-entry vehicle trajectory problem is presented as follows:

1) Assume initial nominal functions $Y(t)$ and $\alpha(t)$.

2) At the $(N+1)$st stage of the iteration $(N=0, 1, 2, \ldots)$, evaluate matrices E and F from Eq. (A10) along the time interval of integration using $Y_N(t)$ and $\alpha_N(t)$.

3) Solve the linear two-point boundary value problem of Eq. (A10) and the boundary conditions in Eqs. (3-6), (A2), and (A4) to obtain $Y_{N+1}(t)$.

4) Compute $\alpha_{N+1}(t)$ from Eq. (A8) using $Y_{N+1}(t)$.

5) Slow down the convergent rate by changing $Y_{N+1}(t)$ to

$$Y_{N+1}(t) = y \ Y_{N+1}(t) + (1-y) \ Y_N(t)$$

and $\alpha_{N+1}(t)$ to

$$\alpha_{N+1}(t) = y\alpha_{N+1}(t) + (1-y)\alpha_N(t)$$

6) Compute ρ as defined in Eq. (32).

7) The optimal solution is obtained when $\rho < \delta$. Otherwise, set $N = N+1$ and go to step 2.

The initial nominal functions employed were

$$\begin{array}{lll} V(t) = 3.0\ \text{ft}'/\text{s}' & & \lambda_1(t) = -0.05 \\ r(t) = 0 & & \lambda_2(t) = 1 \\ h(t) = 0.3\ \text{ft}' & & \lambda_3(t) = -7 \\ x(t) = 0.02t\ \text{ft}' & (0 \le t \le t_f) & \alpha(t) = 0 \end{array}$$

For the first five iterations, $y = 0.3$ used. Subsequently, $y = 1$ was employed. In this study, $\delta = 10^{-3}$ was used.

The optimal solution was obtained after 13 iterations and 460 s of CPU time. CPU time per iteration is 35.4 s. The optimal functions of $\alpha(t)$, sensed deceleration, and $h(t)$ are displayed as dotted lines in Figs. 1-3.

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